Real Numbers: Notes with Detailed Explanation and Examples (One Shot Revision)

Topics to be covered:

- **Fundamental theorem of Arithmetic**
- **Revisiting Rational and irrational numbers**

TOPIC 1: Fundamental Theorem of Arithmetic

Prime and Composite Numbers

A number is called a **prime number** if it has no factors other than 1 and the number itself.
E.g. 2, 3, 5, 7, 11, 23, 43, and 71 are prime numbers.

A number is called a **composite number**, if it has at least one factor other than 1 and the number itself.
E.g. 4, 6, 24, 45, and 188 are composite numbers.

Important Points to Remember:

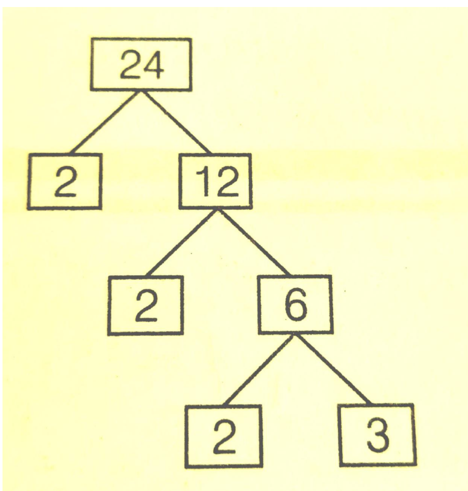
- **1** is neither prime nor composite.
- **2** is the smallest prime number. 2 is the only **even prime number.**
- The smallest composite even number is **4** and the smallest composite odd number is **9.**
- Prime numbers have only two factors, **1 and the number itself**. As 1 is not a prime number, **do not include** it when expressing a number as a product of prime factors.

Factor of a Number:

If a number divides another number exactly (without leaving any remainder), then the number which divides, is called a **factor** of the number that has been divided.
e.g. 1, 2, 3, 6, and 12 are the factors of 24.

Factor Tree

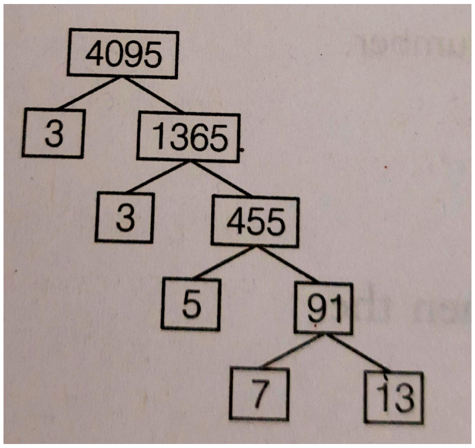
A chain of factors that is represented in the form of a tree, called a **factor tree.**
E.g. Factors of 24 by factor tree are given as



Here, **2 and 3** are prime factors of 6. Number 6 is a factor of 12 and 12 is a factor of 24. This chain of factors is represented in the form of a tree.
Thus, factors of **24 = 2 × 2 × 2 × 3**
24 = $\mathbf{2^3 \times 3}$ as a product of powers of prime.

In this factor tree method, we take the **smallest prime number, which divides the given number exactly, i.e. we try to divide the number by **2 or 3 or 5 or 7 or 11**, and so on.

Example. Factorise the number 4095 through the factor tree.



Hence, 4095 = 3 × 3 × 5 × 7 × 13 as a product of primes.

The Fundamental Theorem of Arithmetic

According to the fundamental theorem of arithmetic, every composite number can be written (factorised) as the **product of primes** and this factorisation expressed is **unique**, apart from the order in which the prime factors occur.

Composite number = **Product of prime numbers**

Or

Any integer greater than 1, either be a prime number or can be written as a unique product of prime numbers.

e.g.15 can be written as **3 × 5 or 5 × 3**, where 3 and 5 are prime numbers.

The prime factorisation of a natural number is **unique**, except for the order of its factors.

e.g. 12 is equal to the product of the prime numbers **2, 2, and 3** together.

i.e. 12 = **2 x 2 x 3** which is same as **3 × 2 × 2** or **2 x 3 x 2.**

We would probably write it as **12 = 2^2 x 3** It is still a **unique combination**(2, 2, and 3).

HCF and LCM By Prime Factorisation Method

The method of finding the **HCF and LCM** of two or more positive numbers by using the **Fundamental theorem of arithmetic** is called the **prime factorisation method.**

In this method, first we express the given two or more numbers as the **product of prime factors** Then,
(i) **HCF** of two or more numbers = **Product of the smallest powe**r of each **common prime facto**r involved in the numbers.
If there is no common prime factor, then the HCF of the given number is 1.

(ii) **LCM** (Least Common Multiple) of two or more numbers is the smallest number that is divisible by all the given numbers.
LCM of two or more numbers = Product of the **greatest power** of each prime factor involved in the numbers, with the highest power.

Example:
Find the LCM and HCF of 120 and 144 by the fundamental theorem of arithmetic.
Solution:
Prime factorisation of 120 = $\mathbf{2 \times 2 \times 2 \times 3 \times 5 = 2^3 \times 3 \times 5}$
and the prime factorisation of 144 = $\mathbf{2 \times 2 \times 2 \times 2 \times 3 \times 3 = 2^4 \times 3^2}$

Here, $\mathbf{2^4, 3^2}$ **and** $\mathbf{5^1}$ are the **greatest exponents** of the prime factors 2, 3 and 5 in two numbers.
Now, LCM (120,144) = $\mathbf{2^4 \times 3^2 \times 5 = 720}$

Here, $\mathbf{2^3}$ **and** $\mathbf{3^1}$ are the **smallest exponent** of the common factors of 2 and 3.
HCF(120, 144) = $\mathbf{2^3 \times 3 = 24}$
Hence, LCM and HCF of 120 and 144 are 720 and 24.

Relation between Numbers and their HCF and LCM:

For any two positive integers a and b, the relation between these numbers and their HCF and LCM is
HCF (a,b) x LCM (a,b) = Product of a and b

HCF (a,b) x LCM (a,b) = a x b

$$\mathbf{HCF(a, b) = \frac{(a \times b)}{LCM(a, b)}}$$

OR

$$\mathbf{LCM(a, b) = \frac{(a \times b)}{HCF(a, b)}}$$

Example 11. If the HCF of two numbers is 12 and their product is 1800, find their LCM.

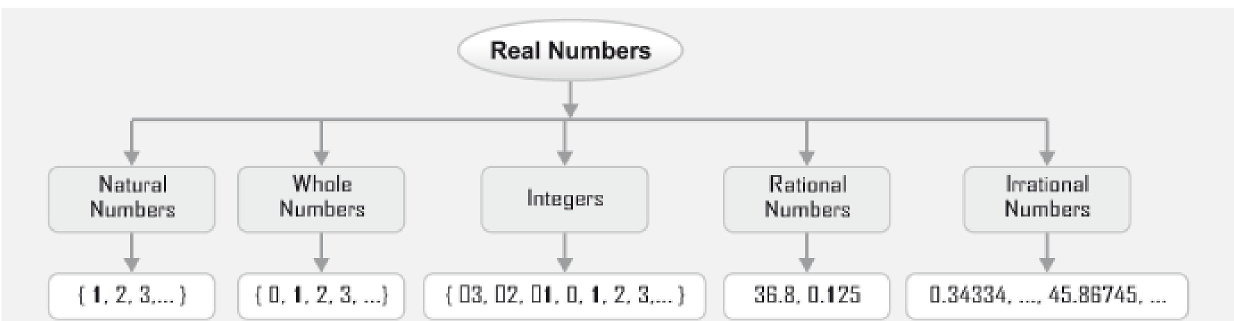
Solution: Let, the two numbers are a and b.
Given: HCF (a, b) = 12 and product (a x b) = 1800
We know that,
HCF (a,b) x LCM (a,b) = Product of a and b
12 x LCM (a, b) = 1800
LCM (a, b) = 1800 ÷ 12
LCM (a, b) = 150
Hence, the required LCM is 150.

Topic 2:
Revisiting Rational and Irrational Numbers

Real Numbers:

A Number which is either rational or irrational is called a real number.

Flow chart of Real numbers:



Rational Numbers

A number that **can be expressed as p/q**, where p and q are integers and q ≠ 0, is called a rational number. e.g.3/7, 13/5, -4/3

Irrational Numbers

A number that **cannot be expressed in the form p/q** where p, and q are integers and q ≠ 0, is called an irrational number.
E.g. √2, √3, √5, 0.2022022202222,..., etc.

Coprime Integers: A pair of integers **having no common factors** other **than 1** (or -1), are called coprime integers.

e.g. (1, 3), (-2, 5), (6, 35), etc are coprime integers.

Theorem 1: Let p be a prime number and a be a positive integer. If p divides a^2, then p divides a.

Theorem 2: $\sqrt{2}$ is irrational.

Proof: Let us assume that $\sqrt{2}$ is rational. Then, it will be of the form where a, and b are integers and $b \neq 0$.
Again, let a and b have no common factor other than 1.
$\sqrt{2} = a/b$, where a and b are **coprime integers**.

On squaring both sides, we get
$2 = a^2/b^2$
$\Rightarrow 2b^2 = a^2$ ……(1)
$\Rightarrow$ 2 divides a^2 $\Rightarrow$ 2 divides a **[from Theorem 1]**
Then, a can be written as 2m, where m is an integer.

On putting a = 2m in Eq. (i), we get
$2b^2 = (2m)^2$
$\Rightarrow 2b^2 = 4m^2$
$\Rightarrow b^2 = 2m^2$
So, 2 divides b^2 $\Rightarrow$ 2 divides a **[from Theorem 1]**
Thus, **2 is a common factor of a and b.**
But this **contradicts** the fact that a and b have no common factor other than 1. The **contradiction** arises by assuming that $\sqrt{2}$ is rational.
Hence, $\sqrt{2}$ is irrational.
Hence proved.

Method of Finding Irrationality of Numbers
We know that a number is an irrational number, if it **cannot be written or expressed in the form p/q,** where p, and q are integers and $q \neq 0$. Infact, for any prime number p, $\sqrt{p}$ is an irrational number.

To prove the irrationality of the numbers, we use the method of **contradiction.**

EXAMPLE: Prove that $\sqrt{3}$ is an irrational number.
We can solve this same like Theorem 2