

Polynomials: Notes with Detailed Explanation and Examples (One Shot Revision)

Topics to be covered:

- Zeroes of a Polynomial
- Relationship between Zeroes and Coefficients of Quadratic Polynomials
- Useful Algebraic Identities

Important Algebraic Identities

$$(i) (a+b)^2 = a^2 + b^2 + 2ab \quad \text{for factorising Polynomials}$$

$$(ii) (a-b)^2 = a^2 + b^2 - 2ab$$

$$(a-b)^2 = (a+b)^2 - 4ab$$

$$(iii) (a+b)^2 - (a-b)^2 = 4ab$$

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$$

$$(iv) (a^2 - b^2) = (a+b)(a-b)$$

$$(v) (a^3 + b^3) = (a+b)(a^2 + b^2 - ab)$$

$$(a^3 + b^3) = (a+b)[(a+b)^2 - 3ab]$$

$$(vi) (a^3 - b^3) = (a-b)(a^2 + b^2 + ab)$$

$$(a^3 - b^3) = (a-b)[(a-b)^2 + 3ab]$$

$$(vii) (a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$(viii) (a+b)^3 = a^3 + b^3 + 3ab(a+b)$$

$$(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$$

$$(ix) (a-b)^3 = a^3 - b^3 - 3ab(a-b)$$

$$(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$$

$$(x) a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

TOPIC 1: Polynomials and Their Related Terms

Polynomials:

A **polynomial** in one variable x , is an algebraic expression of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0$$

where n is a whole number and $a_0, a_1, a_2, \dots, a_n$ are real numbers.

The constants $a_0, a_1, a_2, \dots, a_n$ are also known as **coefficients of polynomials**.

Degree of a Polynomial

The **highest power** of x in a polynomial $p(x)$, is called the **degree** of the polynomial $p(x)$.

e.g. (i) $p(x) = 7x + 2/3$ is a polynomial in variable x of **degree 1** (since, the highest power of x is 1).

(ii) $q(y) = 5y^2 + 5/2y + 9$ is a polynomial in variable y of **degree 2** (since the highest power of y is 2).

Types of Polynomials

1. Linear Polynomial: A polynomial of **degree one**, is called a linear polynomial.

e.g. (i) $f(x) = 4x + 6$

(ii) $p(y) = 3y$

(iii) $f(x) = x - 5/3$

Generally, any linear polynomial in variable x with real coefficients is of form **$f(x) = ax + b$** , where a, b are real numbers and $a \neq 0$.

2. Quadratic Polynomial: A polynomial of **degree two**, is called a quadratic polynomial.

(i) $f(x) = 3x^2 + 5x + 7$

(ii) $f(x) = 5x^2 - 3x$

(iii) $f(x) = -x^2 + \sqrt{5}x$

(iv) $f(y) = 4y^2 + 2/3$

Generally, any quadratic polynomial in variable x with real coefficients is of form $f(x) = ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$.

3. Cubic Polynomial: A polynomial of **degree three**, is called a cubic polynomial.

e.g. (i) $f(z) = 9z^3$

(ii) $f(y) = 13y^3 - 12y^2 + y + 5$ (iii) $f(x) = \sqrt{5} - x^2 + x^3$

Generally, any cubic polynomial in variable x with real coefficients is of form $f(x) = x^3 + bx^2 + cx + d$, where a, b, c, d are real numbers and $a \neq 0$.

4. Biquadratic or Quartic Polynomial: A polynomial of **degree four**, is called a biquadratic polynomial.

e.g.

(i) $f(x) = x^4 + 2x^3 - 7x^2 + 5x + 3$

(ii) $f(z) = z^4 - 7z^2 + 3$

Generally, any biquadratic polynomial in variable x with real coefficients is of form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d, e are real numbers and $a \neq 0$.

Value of a Polynomial at a Given Point

If $p(x)$ is a polynomial in x and α is a real number, then the value obtained by putting $x = \alpha$ in $p(x)$, is called the value of $p(x)$ at $x = \alpha$.

e.g. If $p(x) = 3x^2 - 4x + 1$, then its value at $x = 2$ is

$$p(2) = 3(2)^2 - 4(2) + 1 = 3(4) - 8 + 1 = 12 - 8 + 1 = 5$$

Thus, the value of $p(x)$ at $x = 2$ is 5.

Zero of a Polynomial

A real number k is said to be a zero of a polynomial $p(x)$ if $p(k) = 0$.

e.g. In the given polynomial $p(x) = 4x + 3$, if we put $x = -3/4$, then

$$p(-3/4) = 4 \times (-3/4) + 3 = 0$$

i.e. at $x = -3/4$, $p(x) = 0$,

Therefore $-3/4$ is a zero of a polynomial $p(x) = 4x + 3$

Geometrical Meaning of the Zeroes of a Polynomial

We know that a real number k is a zero of the polynomial $p(x)$, if $p(k) = 0$.

Geometrical Meaning of the Zero of a Linear Polynomial

Consider a **linear polynomial** $ax + b$, $a \neq 0$. We know that, the graph of $y = ax + b$ is a **straight line**, which intersects the **X-axis** at exactly **one point**, namely $(-b/a, 0)$.

Also, we know that the **zero** of $ax + b$ is $-b/a$.

Thus, the zero of $ax + b$ is the **x-coordinate** of the point, where the graph of $y = ax + b$ intersects the X-axis.

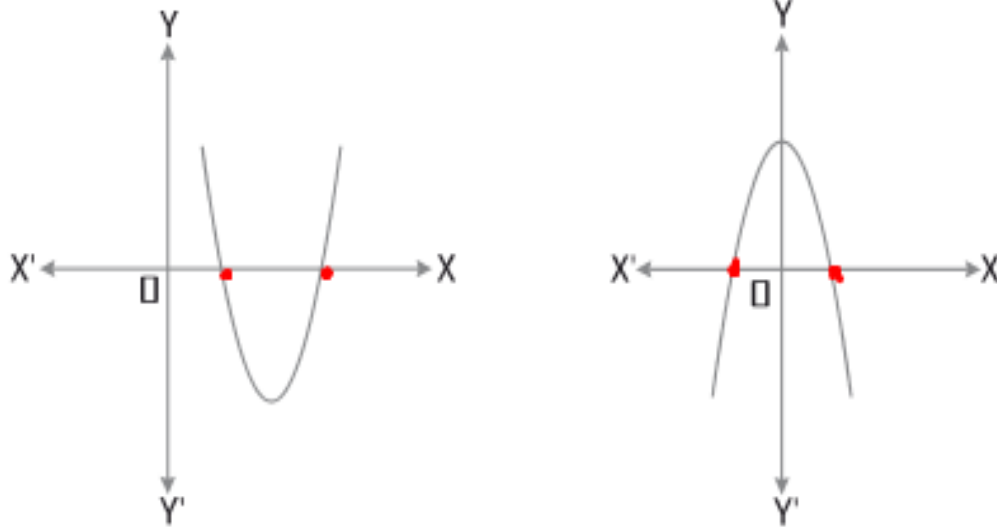
Geometrical Meaning of the Zeroes of a Quadratic Polynomial

For any quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, the graph of the corresponding equation $y = ax^2 + bx + c$ has one of the **two shapes** (which are known as **parabola**) either **open upwards** like  or **open**

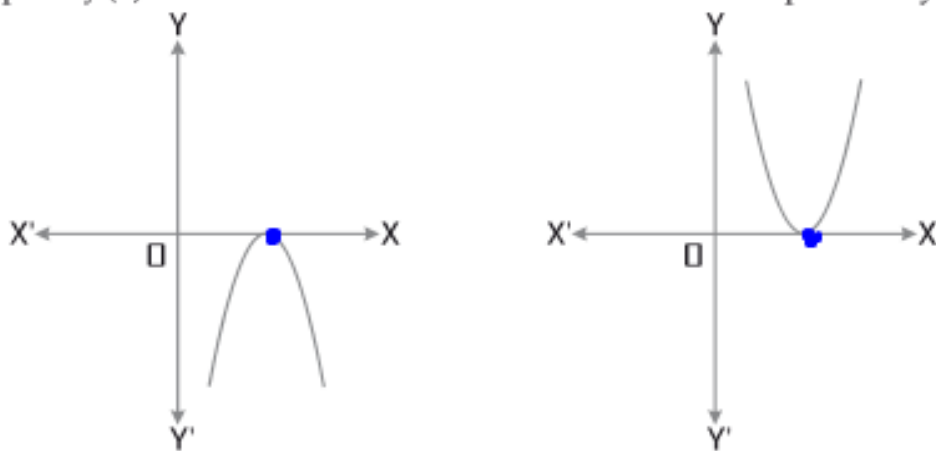
downwards like  depending on whether $a > 0$ or $a < 0$, i.e. if $a > 0$, then shape of $y = ax^2 + bx + c$ is like  and if $a < 0$, then the shape of $y = ax^2 + bx + c$ is like 

As per the observation about the shape of the graph of $y = ax^2 + bx + c$, the following three cases arise:

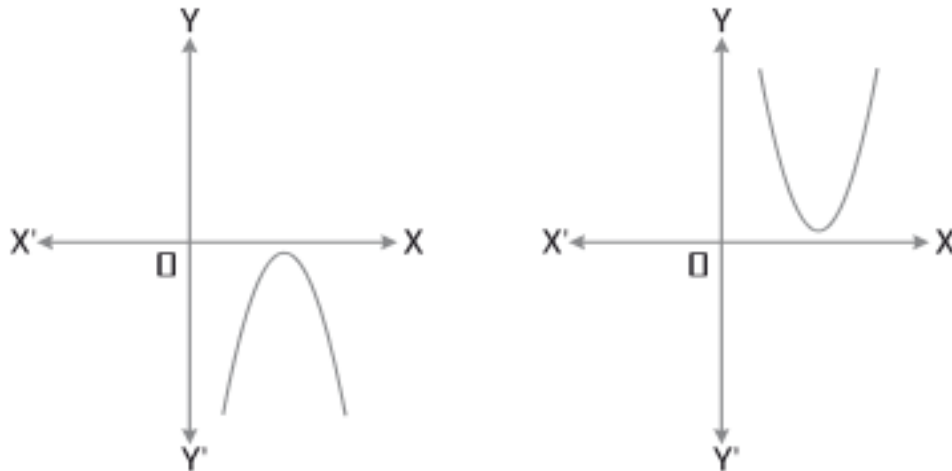
Case I: If the graph intersects the **X-axis at two points**, then the x-coordinate of two points are the **two zeroes** of the quadratic polynomial $ax^2 + bx + c$.



Case II: If the graph intersects or touches the **X-axis at exactly one point**, then the two points (of Case I) coincide. Here, only **one intersecting point** is obtained and the x-coordinate is the **only zero** for the quadratic polynomial $ax^2 + bx + c$.



Case III: If the graph is either **completely above X-axis or completely below X-axis**, i.e. it **does not intersect X-axis at any point**. Then, the quadratic polynomial $ax^2 + bx + c$ has **no zero**.



Hence, geometrically we see that a quadratic polynomial can have either two distinct zeroes or two equal zeroes (i.e. one zero) or no zero.

From this, we can say that a polynomial of **degree 2** has **at most two zeroes.**

In general, the zeroes of a polynomial $p(x)$ are precisely the **x-coordinates** of the points, where the graph of $y = p(x)$ intersects the X-axis.

TOPIC 2: Relationship between Zeroes and Coefficient of a Polynomial, Formation of Quadratic and Cubic Polynomials

Relationship between Zeroes and the Coefficient of a Polynomial:

Relationship between the zeroes and the coefficients of a Polynomial :

S. No.	Type of polynomial	General form	Maximum Number of zeroes	Relationship between zeroes and coefficients
1.	Linear	$ax + b; a \neq 0$	1	$k = -\frac{b}{a}, \text{ i.e., } k = -\frac{\text{Constant term}}{(\text{Coefficient of } x)}$
2.	Quadratic	$ax^2 + bx + c; a \neq 0$	2	Sum of zeroes $(\alpha + \beta)$ $= -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\frac{b}{a}$ Product of zeroes $(\alpha\beta)$ $= \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{c}{a}$
3.	Cubic	$ax^3 + bx^2 + cx + d; a \neq 0$	3	Sum of zeroes $(\alpha + \beta + \gamma)$ $= -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3} = -\frac{b}{a}$ Product of sum of zeroes taken two at a time $(\alpha\beta + \beta\gamma + \gamma\alpha)$ $= +\frac{\text{Coefficient of } x}{\text{Coefficient of } x^3} = \frac{c}{a}$ Product of zeroes $(\alpha\beta\gamma)$ $= -\frac{\text{Constant term}}{\text{Coefficient of } x^3} = -\frac{d}{a}$

Formation of Quadratic and Cubic Polynomials

(i) If α and β are the zeroes of a quadratic polynomial, then the quadratic polynomial will be

$x^2 - (\text{sum of zeroes})x + \text{product of zeroes}$

i.e. $x^2 - (\alpha + \beta)x + \alpha\beta$

where, sum of zeroes = $(\alpha + \beta)$ and product of zeroes = $\alpha\beta$.

E.g If the sum and product of the zeroes of quadratic polynomials are 5 and 7 respectively, then the quadratic polynomial will be $x^2 - 5x + 7$.

(ii) If α , β and γ are the zeroes of the cubic polynomial, then the cubic polynomial will be

$x^3 - (\text{sum of zeroes})x^2 + (\text{sum of the product of zeroes taking two at a time})x - \text{product of zeroes}$

ie. $x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$

where, the sum of zeroes = $\alpha + \beta + \gamma$, the sum of the product of two zeroes taken at a time = $\alpha\beta + \beta\gamma + \gamma\alpha$ and the product of zeroes = $\alpha\beta\gamma$.

e.g. If the sum of zeroes, the sum of the product of zeroes taken two at a time, and the product of all zeroes are -2, 3, and 5 respectively, then the cubic polynomial will be

$x^3 - (-2)x^2 + 3x - 5$.

Any other quadratic polynomial which also satisfies the given conditions will be of the form $k(x^2 + 5x + 2)$, where k is a real number.